Duality and Boundary Yangians[12]

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Abstract

The existence of dual structures in a Yangian $Y(g)$ signify that the latter belongs to multidimensional naturally parametrized variety of Hopf algebras. These varieties have boundaries containing Yangians $Y(a)$ inequivalent to the original $Y(g)$. The new basic algebra a is an algebra of the cotangent bundle attributed to the dual structure. We show how to construct such boundary Yangians and study some of their properties. We prove that the Hopf algebra $Y(a)$ is a quantization of a parametric solution of the classical Yang-Baxter equation. The limiting procedure and the properties of the boundary Yangians are demonstrated explicitly for the case of $Y(sl(2))$.

1 Introduction

The solutions of Yang-Baxter equation (YBE) depending on spectral parameter are of special importance in mathematical physics. They are connected with Wess-Zumino-Witten models and affine Toda field theories [1],[2]. The algebraic base of these theories is formed by the universal enveloping algebra $U(g[u])$ of polynomial current algebra $g[u]$ with g being a finite-dimensional complex Lie algebra.

Yangians $Y(g)$ were introduced by Drinfeld [3] as a quantum deformations of the algebra $U(g[u])$. They correspond to the rational solutions of the classical Yang-Baxter equation (CYBE) first found by Belavin [4] and completely described by Belavin and Drinfeld [5]. The classification of rational solutions of CYBE for simple Lie algebras was performed by Stolin [6].

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The quantum deformation $\mathcal{R}(\lambda)$ of such rational solution

$$r(\mu-\nu)=\frac{C^2}{\mu-\nu} \tag{1}$$

(with C^2 being the second order Casimir element of the algebra g) satisfies the parametric Yang-Baxter equation,

$$\mathcal{R}_{12}(\lambda_1-\lambda_2)\mathcal{R}_{13}(\lambda_1-\lambda_3)\mathcal{R}_{23}(\lambda_2-\lambda_3)=\mathcal{R}_{23}(\lambda_2-\lambda_3)\mathcal{R}_{13}(\lambda_1-\lambda_3)\mathcal{R}_{12}(\lambda_1-\lambda_2),$$

and realizes the morphism to the Yangian with the opposite comultiplication:

$$(T_\lambda\otimes \mathrm{id})\,\Delta^{\mathrm{op}}(a)=\mathcal{R}(\lambda)\left((T_\lambda\otimes \mathrm{id})\Delta(a)\right)(\mathcal{R}(\lambda))^{-1}, \tag{2}$$

where T_λ is a parameter shifting operator.

In this report we consider the problem of constructing the quantizations of $U(a[u])$ where the algebra a is not semisimple and also present rational solutions of CYBE for such algebras.

Our approach is based on studying the properties of the boundaries of parametrized sets of Yangians. We find that under certain conditions the corresponding algebraic constructions survive on the boundaries of the parametrized domain and can be explicitely described. To reach the boundaries we use the restricted limiting procedure based on the existence of dual structure in the corresponding Hopf algebra H. The existence of a dual structure in H means that there exists the two-dimensional set of Hopf algebras containing H. One of the properties of such sets is that when the corresponding parameters go to its limiting values the algebraic construction survives. In most of the cases the Hopf algebra thus obtained is inequivalent to the original one. This was demonstrated for deformation quantizations of the finite-dimensional Lie algebras [7]. Now we show that the same is true also for Yangians and other possible quantizations of $U(g[u])$.

2 Yangian $Y(sl(2))$ and its natural limits

The Yangian $Y(sl(2))$ is a Hopf algebra generated by the elements $\{e_k,h_k,f_k\}$ ($k\in\mathbf{Z}_+$) with relations

$$\begin{array}{lcl}
[h_k,h_l]=0, \qquad [e_k,f_l]=h_{k+l}, & & \\
{[h_0,e_l]}=2e_l, \qquad [h_0,f_l]=-2f_l, & & \\
{[h_{k+1},e_l]}-[h_k,e_{l+1}] & = & \hbar\{h_k,e_l\}, \\
{[h_{k+1},f_l]}-[h_k,f_{l+1}] & = & -\hbar\{h_k,f_l\}, \\
{[e_{k+1},e_l]}-[e_k,e_{l+1}] & = & \hbar\{e_k,e_l\}, \\
{[f_{k+1},f_l]}-[f_k,f_{l+1}] & = & -\hbar\{f_k,f_l\},
\end{array} \tag{3}$$

where $\hbar$ is the deformation parameter. The coproducts for the generators of $sl(2) \in Y(sl(2))$ rest primitive:

$$\Delta(x) = x \otimes 1 + 1 \otimes x; \quad x \in sl(2), \tag{4}$$

while the nontrivial coalgebraic part is uniquely defined by the following comultiplications (and the multiplications (3) above):

$$\begin{array}{rcl} \Delta(e_1) & = & e_1 \otimes 1 + 1 \otimes e_1 + \hbar h_0 \otimes e_0, \\ \Delta(f_1) & = & f_1 \otimes 1 + 1 \otimes f_1 + \hbar f_0 \otimes h_0. \end{array} \tag{5}$$

To be able to study general properties of Yangian's Hopf structure the generating functions formalism is especially convenient. In terms of

$$\begin{array}{l} e(u) := \sum_{k \geq 0} e_k u^{-k-1}, \;\; f(u) := \sum_{k \geq 0} f_k u^{-k-1}, \\ h(u) := 1 + \hbar\chi(u) := 1 + \hbar \sum_{k \geq 0} h_k u^{-k-1} \end{array} \tag{6}$$

the compositions generated by (3,4,5) look like

$$\begin{array}{ll} [h(u), h(v)] = 0, & [e(u), f(v)] = -\frac{1}{\hbar}\frac{h(u)-h(v)}{u-v}, \\ [h(u), e(v)] = -\hbar\frac{\{h(u),(e(u)-e(v))\}}{u-v}, & [h(u), f(v)] = \hbar\frac{\{h(u),(f(u)-f(v))\}}{u-v}, \\ [e(u), e(v)] = -\hbar\frac{(e(u)-e(v))^2}{u-v}, & [f(u), f(v)] = \hbar\frac{(f(u)-f(v))^2}{u-v}. \end{array} \tag{7}$$

The coproducts for the generating functions are written in the form proposed by Molev [8]

$$\begin{array}{rcl} \Delta(e(u)) & = & e(u) \otimes 1 + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} (f(u+\hbar))^k h(u) \otimes (e(u))^{k+1}, \\ \Delta(f(u)) & = & 1 \otimes f(u) + \sum_{k=0}^{\infty} (-1)^k \hbar^{2k} (f(u))^{k+1} \otimes h(u)(e(u+\hbar))^k, \\ \Delta(h(u)) & = & \sum_{k=0}^{\infty} (-1)^k (k+1) \hbar^{2k} (f(u+\hbar))^k h(u) \otimes h(u)(e(u+\hbar))^k. \end{array} \tag{8}$$

The dual structure in $Y(sl(2))$ is inherited from the basic subalgebra $sl(2)$ interpreted as a classical double. When the double structure is in its prime form the dual parameters and the corresponding analytical family of Hopf algebras can be canonically introduced (see [9]). In the case of Yangian this double is factorized. To get the necessary parametrization we must reconstruct the prefactorized relations. Such reconstruction can be achieved if one takes into account that the unfactorized classical double of the initial $sl(2)$ subalgebra has the form

$$\begin{array}{ll} [h, h'] = 0, & [e, f] = \frac{1}{2}(h + h'), \\ [h, e] = 2e, & [h, f] = -2f, \\ [h', e] = 2e, & [h', f] = -2f. \end{array} \tag{9}$$

This means that the transformation

$$\begin{array}{rcl} e_k & \to & \frac{1}{p} e_k, \\ f_k & \to & \frac{1}{t} f_k, \\ h_k & \to & \frac{1}{2}\left(\frac{h_k}{p} + \frac{h'_k}{t}\right) \end{array} \tag{10}$$

accompanied by the rescaling of the original $sl(2)$ structure constants will lead to the parametrization that would not be canonical in the whole two-dimensional domain but might work well just in the neighborhood of its boundaries. According to the general rules [9] the deformation parameter must be also rescaled:

$$\hbar \to pt. \tag{11}$$

All these transformations and rescalings produce the parametrized algebraic construction $Y_{pt}(sl(2))$ well defined only when one of the parameters (p or t) is small.

When a deformation quantization algebra admits a canonical parametrization (according to some dual structure) its quasiclassical limits can be found. In our case the limits $Y_{p,0}(a) := \lim_{t\to 0} Y_{pt}(sl(2))$ and $Y_{0,t}(b) := \lim_{p\to 0} Y_{pt}(sl(2))$ are equivalent due to the selfduality of the Borel subalgebra in the classical double algebra (9). Thus we need to consider only one of these algebraic constructions, that we call boundary Yangians. The limiting procedure gives the following structure constants for the boundary Yangian $Y_{p,0}(a)$:

$$\begin{array}{lll} [h_k, h_l] = 0, & [h'_k, h'_l] = 0, & [h_k, h'_l] = 0, \\ [h_0, e_l] = 4pe_l, & [h_0, f_l] = -4pf_l, & \\ [h'_0, e_l] = 0, & [h'_0, f_l] = 0, & [e_k, f_l] = \frac{p}{2} h'_{k+l}, \\ [h_{k+1}, e_l] - [h_k, e_{l+1}] & = & p^2 \{h'_k, e_l\}, \\ [h_{k+1}, f_l] - [h_k, f_{l+1}] & = & -p^2 \{h'_k, f_l\}, \\ [h'_{k+1}, e_l] - [h'_k, e_{l+1}] & = & 0, \\ [h'_{k+1}, f_l] - [h'_k, f_{l+1}] & = & 0, \\ [e_{k+1}, e_l] - [e_k, e_{l+1}] & = & 0, \\ [f_{k+1}, f_l] - [f_k, f_{l+1}] & = & 0, \end{array} \tag{12}$$

The coproducts for the "zero mode" generators e_0, f_0, h_0, h'_0 rest primitive. The others are defined by the relations:

$$\begin{array}{rcl} \Delta(e_1) & = & e_1 \otimes 1 + 1 \otimes e_1 + \frac{p}{2} h'_0 \otimes e_0, \\ \Delta(f_1) & = & f_1 \otimes 1 + 1 \otimes f_1 + \frac{p}{2} f_0 \otimes h'_0, \\ \Delta(h_1) & = & h_1 \otimes 1 + 1 \otimes h_1 + \frac{p}{2}(h'_0 \otimes h_0 + h_0 \otimes h'_0) \\ & & -4pf_0 \otimes e_0, \\ \Delta(h'_1) & = & h'_1 \otimes 1 + 1 \otimes h'_1 + \frac{p}{2}(h'_0 \otimes h'_0). \end{array} \tag{13}$$

The internal structure of this Hopf algebra becomes more transparent in terms of generating functions:

$$
\begin{array}{lll}
[\chi(u),\chi(v)] = 0, & [\chi'(u),\chi'(v)] = 0, & [\chi(u),\chi'(v)] = 0, \\
[\chi'(u),e(v)] = 0, & [\chi'(u),f(v)] = 0, & \\
[e(u),f(v)] = -\frac{p}{2}\frac{\chi'(u)-\chi'(v)}{u-v}, & & \\
[\chi(u),e(v)] \;=\; -\frac{p}{u-v}\{2+p\chi'(u),e(u)-e(v)\} & & \\
[\chi(u),f(v)] \;=\; \frac{p}{u-v}\{2+p\chi'(u),f(u)-f(v)\} & &
\end{array}
\tag{14}
$$

$$
\begin{array}{rcl}
\Delta(e(u)) & = & e(u)\otimes 1 + 1\otimes e(u) + \frac{p}{2}\chi'(u)\otimes e(u), \\
\Delta(f(u)) & = & f(u)\otimes 1 + 1\otimes f(u) + \frac{p}{2}f(u)\otimes \chi'(u), \\
\Delta(\chi'(u)) & = & \chi'(u)\otimes 1 + 1\otimes \chi'(u) + \frac{p}{2}(\chi'(u)\otimes \chi'(u)), \\
\Delta(\chi(u)) & = & \chi(u)\otimes 1 + 1\otimes \chi(u) + \\
 & & +\frac{p}{2}(\chi'(u)\otimes\chi(u) + \chi(u)\otimes\chi'(u)) \\
 & & -4pf(u)(1+\frac{p}{2}\chi'(u))\otimes(1+\frac{p}{2}\chi'(u))e(u),
\end{array}
\tag{15}
$$

where $e(u), f(u)$ and $\chi(u)$ are as in (6) while $\chi'(u)$ is the analog of $\chi(u)$ for the generator h'. This Hopf algebra $Y_{p,0}(a)$ is a Yangian for a nonsemisimple Lie algebra a with the following compositions:

$$
\begin{array}{c}
[h_0', h_0] = 0, \\
[h_0, e_0] = 4pe_0, \qquad [h_0', e_0] = 0, \\
[h_0, f_0] = -4pf_0, \quad [h_0', f_0] = 0, \\
[e_0, f_0] = \frac{p}{2}h_0',
\end{array}
\tag{16}
$$

This is just the cotangent bundle algebra for the two-dimensional Borel subalgebra of sl(2).

For possible applications it is crutially important to know whether the Yangians like $Y_{p,o}(a)$ are pseudotriangular or not, that is whether they have the universal $\mathcal{R}$-matrix providing the property (2). Below we shall present the indications that such $\mathcal{R}$-matrises exist.

The direct way to solve the problem is to construct the parametrized version of the expansion terms for $\mathcal{R}$-matrix in the general case of $Y_{p,t}(sl(2))$ and to check their limits. In our case this aproach doesn't work. The first nontrivial term – the classical r-matrix after being parametrized according to the transformation (10) diverges in the limit point. As we have seen above the limiting procedure must be accompanied by the rescaling of commutators. This clearly indicates that certain terms of the r-matrix must be rescaled in the neighborhood of the limit point. It is not difficult to find these terms and to perform an adequate rescaling. The result is formulated in the statement that follows.

<u>Theorem.</u> *The boundary Yangian $Y_{p,0}(a)$ (correspondingly $Y_{0,t}(a)$) originating from $Y(sl(2))$ is the deformation quantization of the polinomial current*

algebra based on the algebra a (16). The first order expansion term for this deformation is defined by the following solution of the classical Yang-Baxter equation for the algebra a:

$$r(u,v) = \frac{1}{u-v}\left[\frac{1}{8}(h_0 \otimes h_0' + h_0' \otimes h_0) + e_0 \otimes f_0 + f_0 \otimes e_0\right]. \qquad (17)$$

The first assertion becomes obvious when the classical limit of $Y_{p,0}(a)$ is considered. The validity of the last statement can be checked by the direct computation of the dual Lie algebra defined by (17); it coinsides with the co-Lie structure that one observes in (13) or (15).

To make the demonstration most transparent we have studied the simpliest possible case – the Yangian based on $sl(2)$ algebra. As a result our boundary Yangian $Y(a)$ admits some additional simplifications. The subalgebra generated by $\chi'(u)$ forms a Hopf ideal $J(\chi'(u)) \in Y(a)$. In the factor algebra $\frac{Y(a)}{J(\chi'(u))} \equiv Y(c)$ the only nontrivial relations are

$$\begin{array}{rcl} [\chi(u), e(v)] & = & -\frac{4p}{u-v}(e(u)-e(v)), \\ [\chi(u), f(v)] & = & \frac{4p}{u-v}(f(u)-f(v)) \end{array} \qquad (18)$$

and

$$\Delta(\chi(u)) = \chi(u) \otimes 1 + 1 \otimes \chi(u) - 4pf(u) \otimes e(u). \qquad (19)$$

This Yangian $Y(c) \approx U_{\mathcal{F}}(c[u])$ is a quantized algebra of currents for the Lie algebra c, where c is the algebra $(sl(2))^{\mathrm{contr}}$ with the trivially contracted composition $[e,f]=0$.

The defining relations show that the multiplications in $Y(c)$ are undeformed; the commutators are classical and coinside with those of $U(c[u])$,

$$\begin{array}{rclrcl} [h_k, h_l] & = & 0; & [h_0, e_l] & = & 4pe_l; \\ [e_k, f_l] & = & 0; & [h_0, f_l] & = & -4pf_l; \\ [e_{k+1}, e_l] & = & [e_k, e_{l+1}]; & [f_{k+1}, f_l] & = & [f_k, f_{l+1}]; \\ [h_{k+1}, e_l] & = & [h_k, e_{l+1}]; & [h_{k+1}, f_l] & = & [h_k, f_{l+1}]. \end{array} \qquad (20)$$

It also has trivial coproducts for $e_k,\ f_k \quad (k \in \mathbf{Z}^+)$ and h_0. The nontrivial costructure in $Y(c)$ is generated by the relation

$$\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1 - 4pf_0 \otimes e_0$$

and the compositions (20).

This quantized algebra can be obtained from $U(c[u])$ by a twisting procedure: $U(c[u]) \xrightarrow{\mathcal{F}} Y(c)$. The carrier of this twist is an abelian subalgebra generated by the primitive elements e_0 and f_0 and the twisting element has the form

$$\mathcal{F}(u-v) = \exp\left(\frac{1}{v-u} f_0 \otimes e_0\right).$$

This gives the possibility to write down the universal element for $Y(c)$,

$$\mathcal{R}(u-v) = \exp\left(\frac{1}{u-v}(f_0 \otimes e_0 + e_0 \otimes f_0)\right).$$

It can be checked that this $\mathcal{R}$-matrix provides the pseudotriangularity of $Y(c)$. The first nontrivial term of the power series expansion of $\mathcal{R}$ coinsides with the classical r-matrix that can be obtained from the expression (17) after the factorization by the ideal $J(h'_k)|_{k\in\mathbf{Z}^+}$.

3 Conclusions

We have demonstrated that dual structures signify the existence of the nontrivial limiting algebraic objects corresponding to the rational solutions of CYBE based on nonsemisimple Lie algebras. Thus rational solutions of the type (1) are not necessarily connected with the nondegeneracy of the Killing form.

The existence of such rational solutions for the so called symmetric algebras was established in [11]. It was also mentioned there that probably Yangians for symmetric algebras exist. Our result proves this supposition. An important subclass of symmetric algebras is presented by Manin triples. These are the algebras $(a, b(2), b(2)^*)$ (with two-dimensional Borel algebra $b(2)$) that form a Manin triple in our case and its characteristic nondegenerate form was used to construct the invariant element in (17).

The procedure proposed above is quite general and can be applied to any Hopf algebra that have the form of deformation quantization. For example, it can be applied to the Yangian double (DY) where the dual structure is more simple than in the case of the Yangian itself – the double is not factorized in DY. It must be noted that in this case the situation with the canonical limits for the $\mathcal{R}$-matrix is complicated by the specific form of dualization established for DY [10]. It depends on the quantization parameter and fails in the canonical limits. To reobtain quasitriangularity for the corresponding boundary Yangian one must reformulate the structures of the Yangian double in terms of the canonical dualization.

It is significant that after the factorization the boundary Yangian can be presented in the form of a twist so that its $\mathcal{R}$-matrix can be written explicitly.

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References

[1] D. Bernard, Commun. Math. Phys. **137**, 191-208 (1991).

[2] D. Bernard, A. LeClair, Commun. Math. Phys. **142**, 99-138 (1991).

[3] V.G. Drinfeld, Soviet Math. Docl., **32**, 254-258 (1985); **36**, 212-216 (1988).

[4] A.A. Belavin, Functional. Anal. Appl. **14**, 18-26 (1980).

[5] A.A. Belavin, V.G.Drinfeld, Funct. Anal. Appl. **16**, 159-180 (1982).

[6] A. Stolin, Commun Math. Phys. **141**, 533-548 (1991).

[7] V.D. Lyakhovsky, V.I.Tkach, Journ. Phys. A **31**, 2869-2880 (1998).

[8] A.I. Molev, Journ. Math. Phys. **36**, 923-945 (1995).

[9] V.D. Lyakhovsky, A.M. Mirolyubov, Intern. Journ. Mod. Phys. A, **12**, 225 (1997).

[10] S. Khoroshkin, V. Tolstoy, Lett. Math. Phys. **36**, 373-402, (1996).

[11] K. Beidar, Y. Fong, A. Stolin, Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl. 46, 15-28 (1997).